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The Hahn and Meixner polynomials of an imaginary argument and some of their applications

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Abstract. The Hahn and Meixner polynomials belonging to the classical orthogonal polynomials of a discrete variable are analytically continued in the complex plane both in variable and parameter. This leads to the origination of two systems of real polynomials orthogonal with respect to a continuous measure. The Meixner polynomials of an imaginary argument obtained in this manner turned out to be known in the literature as the Pollaczek polynomials. The orthogonality relation for the Hahn polynomials with respect to a continuous measure is apparently new. A close connection between the Hahn polynomials of an imaginary argument and representations of the Lorentz group SO(3, 1) is considered.

1. Introduction

Among the special functions of mathematical physics an important place belongs to the *classical orthogonal polynomials of a discrete variable* which are the difference analogues of the Jacobi, Laguerre and Hermite polynomials on the uniform and non-uniform lattices. Recently (see Nikiforov *et al* 1982, Nikiforov and Ouvarov 1983, Nikiforov and Uvarov 1983, Nikiforov *et al* 1984) a simple approach to the theory of these polynomials has been developed which allows us to single out naturally the above-mentioned class of special functions, to derive in a simple way all their main properties and to carry out their classification. The Hahn, Meixner, Krawtchouk and Charlier polynomials (see, for example Bateman and Erdelyi 1953, Szego 1959, Nikiforov and Ouvarov 1983, Nikiforov *et al* 1984), as well as the polynomials introduced by Hahn (1949), Karlin and McGregor (1961), Askey and Wilson (1979) and Wilson (1980) with different special considerations, proved to be the particular cases of the classical orthogonal polynomials of a discrete variable.

In the present paper a simple technique is discussed which permits one to include in the general theory of classical orthogonal polynomials of a discrete variable some more important families. As is well known, the Hahn, Meixner, Krawtchouk and Charlier polynomials are orthogonal on a discrete set of points. If we write a discrete orthogonality relation for these polynomials in the form of a contour integral using Cauchy's theorem and subsequently open up the contour in the complex plane, then in some cases after analytic continuation in the parameter on a line, parallel to the imaginary axis, there arises a real system of polynomials orthogonal with respect to a continuous measure. It is natural to call such polynomials the *classical orthogonal polynomials of a discrete variable of the imaginary argument.* The transition from the discrete orthogonality property of these polynomials to the continuous one is analogous to the well known Sommerfeld–Watson transformation in optics and quantum theory of scattering.

We shall discuss a new orthogonality property of the Hahn and Meixner polynomials with respect to a continuous measure. As is demonstrated in § 2, the Meixner polynomials of an imaginary argument are the Pollaczek polynomials (Pollaczek 1949a, b, 1950a, b), traditionally considered as a special case in the theory of orthogonal polynomials (Bateman and Erdelyi 1953, Szego 1959). In § 3 the orthogonality property of the Hahn polynomials in a continuous variable is established which apparently has not been encountered in the literature for the general case.

The classical orthogonal polynomials of a discrete variable are used in various problems of theoretical and mathematical physics, group representation theory, computational physics and techniques. It is sufficient to mention, for instance, the application of these polynomials in the quantum theory of angular momentum or in the SU(2) group representation theory. Recently close relationships have been found between generalised spherical harmonics for SU(2) and the Krawtchouk polynomials (Koornwinder 1982), between the Clebsch–Gordan coefficients for SU(2) and the Hahn polynomials (Gel'fand *et al* 1958, Ryvkin 1959, Meckler 1959, Kirichenko and Stepanovsky 1974, Koornwinder 1981, Smorodinsky and Suslov 1982a, Nikiforov and Suslov 1982, Nikiforov *et al* 1983a, b), between Wigner $6j$-symbols and the Racah polynomials (Wilson 1980, Smorodinsky and Suslov 1982b, Suslov 1983a, Nikiforov *et al* 1983a, b). From such a viewpoint, the quantum theory of angular momentum becomes even more complete and logically consistent. The above results are generalised for the discrete positive series of the unitary irreducible representations for the non-compact group SU(1, 1) (Smirnov *et al* 1984). The T coefficients of the method of trees can be expressed through the same polynomials (Suslov 1983b).

In the present work some further applications are also discussed. In § 4 the wavefunctions in a quasipotential model of a linear relativistic oscillator (Atakishiyev *et al* 1980) are expressed through the Pollaczek polynomials (Atakishiyev 1983, 1984). In § 5 close connections between the unitary irreducible representations of the Lorentz group SO(3, 1) and the Hahn polynomials of an imaginary argument (Suslov 1984a) are discussed.

2. The Pollaczek polynomials as the Meixner polynomials of an imaginary argument

The *Meixner polynomials* $m_n^{(\gamma,\mu)}(z)$ belong to the classical orthogonal polynomials of a discrete variable whose properties have been well studied (Bateman and Erdelyi 1953, Szego 1959, Nikiforov and Ouvarov 1983, Nikiforov *et al* 1984). The Meixner polynomials may be defined by means of the three-term recurrence relation:

$$\mu m_{n+1}^{(\gamma,\mu)}(z)=[\gamma\mu+(1+\mu)n-(1-\mu)z]m_n^{(\gamma,\mu)}(z)-n(n+\gamma-1)m_{n-1}^{(\gamma,\mu)}(z) \tag{2.1}$$

with the initial conditions $m_0^{(\gamma,\mu)}(z)=1$ and $m_{-1}^{(\gamma,\mu)}(z)=0$. For real values of the parameters $\gamma>0$ and $0<\mu<1$ they satisfy the discrete orthogonality relation

$$\sum_{k=0}^{\infty} m_n^{(\gamma,\mu)}(k)m_{n'}^{(\gamma,\mu)}(k)\rho(k)=\frac{n!\,(\gamma)_n}{\mu^n(1-\mu)^\gamma}\,\delta_{nn'} \tag{2.2}$$

where $\rho(k)=\mu^k(\gamma)_k/k!$ and $(\gamma)_z=\Gamma(\gamma+z)/\Gamma(\gamma)$.

We consider the analytic continuation of the relation (2.2) in the parameter μ, leading to the polynomials orthogonal with respect to a continuous measure. Using Cauchy's theorem the left-hand side of (2.2) may be written as an integral with a contour C_1, enclosing the positive real axis Re $z>0$ (figure 1), namely

$$\frac{1}{2\pi i}\int_{C_1} m_n^{(\gamma,\mu)}(z)m_{n'}^{(\gamma,\mu)}(z)\tilde{\rho}(z)\,\mathrm{d}z=\frac{n!\,(\gamma)_n}{\mu^n(1-\mu)^\gamma}\delta_{nn'} \tag{2.3}$$

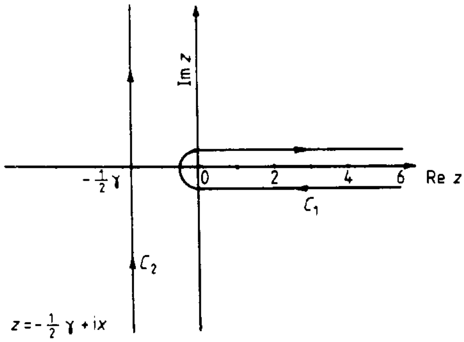

Figure 1.

where $\tilde{\rho}(z)=(\gamma)_z\Gamma(-z)(-\mu)^z$. Using the asymptotic behaviour of the gamma function in the complex z plane it is possible to show that on the semicircle $z=-\gamma/2+R\,\mathrm{e}^{\mathrm{i}\theta}$ and $-\pi/2\leqslant\theta\leqslant\pi/2$ for the function $\tilde{\rho}(z)$ the estimate

$$\tilde{\rho}(z)=\mathrm{O}(R^{\gamma-1}\exp\{R[\cos\theta\ln|\mu|-\sin\theta(\arg(-\mu)\pm\pi)]\}) \tag{2.4}$$

does hold as $R\to\infty$. Therefore for $|\mu|<1$ and $|\arg(-\mu)|<\pi$ the integration contour C_1 in (2.3) can be replaced by the contour C_2, on which $z=-\gamma/2+\mathrm{i}x$, $-\infty<x<\infty$. According to the estimate (2.4) for the function $\tilde{\rho}(z)$, when $|\arg(-\mu)|<\pi$ the integral in the relation (2.3) uniformly converges on the contour C_2, where $\theta=\pm\pi/2$, for all values of $|\mu|$†. Consequently, this integral can be analytically continued in the parameter μ to the entire complex μ plane with the cut along the positive real axis Re $\mu>0$. In particular, the equality (2.3) remains valid for both $\mu=\exp(-2\mathrm{i}\varphi)$ and $z=-\gamma/2+\mathrm{i}x$ ($\gamma>0, 0<\varphi<\pi$). If we set (see Atakishiyev 1983, 1984, Suslov 1984a)

$$P_n^\lambda(x,\varphi)=\frac{\exp(-\mathrm{i}n\varphi)}{n!}m_n^{(2\lambda,\mu)}(-\lambda+\mathrm{i}x)\qquad\mu=\exp(-2\mathrm{i}\varphi) \tag{2.5}$$

then (2.1) leads to a three-term recurrence relation with the real coefficients for the polynomials $P_n^\lambda(x,\varphi)$ with $\lambda>0$ and $0<\varphi<\pi$, i.e. they are real for the real values of the variable x. As follows from (2.3) and (2.5) the polynomials $P_n^\lambda(x,\varphi)$ satisfy the orthogonality relation

$$\int_{-\infty}^{\infty}P_n^\lambda(x,\varphi)P_{n'}^\lambda(x,\varphi)\rho(x)\,\mathrm{d}x=\delta_{nn'}\Gamma(2\lambda+n)/n! \tag{2.6}$$

† It will be recalled that to prove the regularity of an integral $F(\mu)=\int_C f(z,\mu)\,\mathrm{d}z$, depending on the parameter μ, belonging to a domain D of the complex μ plane, the uniform convergence of this integral with respect to $\mu\in D'$ is required, where D' is any closed subdomain of D (see, for example, Efgrafov 1968).

with respect to a continuous measure with the weight†

$$\rho(x)=\frac{1}{2\pi}(2\sin\varphi)^{2\lambda}|\Gamma(\lambda+\mathrm{i}x)|^2\exp[(2\varphi-\pi)x]. \tag{2.7}$$

The polynomials $P_n^\lambda(x,\varphi)$ were introduced previously by Pollaczek (1949a, b, 1950a, b) and called the Pollaczek polynomials. We have demonstrated that Pollaczek polynomials are the analytic continuations of the Meixner polynomials in the parameter μ.

Using the well studied properties of the Meixner polynomials (see, for example, Nikiforov *et al* 1984) and (2.5), it is easy to obtain the difference equation, Rodrigues' formula, etc, for the Pollaczek polynomials. In particular, the difference equation has the form

$$\begin{aligned}&[(\lambda-\mathrm{i}x)\exp[\mathrm{i}(\varphi+\partial_x)]-(\lambda+\mathrm{i}x)\exp[-\mathrm{i}(\varphi+\partial_x)]P_n^\lambda(x,\varphi)\\&\qquad=2\mathrm{i}[(n+\lambda)\sin\varphi-x\cos\varphi]P_n^\lambda(x,\varphi)\end{aligned} \tag{2.8}$$

where $\partial_x=\mathrm{d}/\mathrm{d}x$ and $\exp(\alpha\partial_x)f(x)=f(x+\alpha)$. We would also like to mention the formulae for the action of the raising and lowering operators:

$$\begin{aligned}&\{(n+\lambda+\mathrm{i}x)\exp(-\mathrm{i}\varphi)+(\lambda-\mathrm{i}x\}\exp[\mathrm{i}(\varphi+\partial_x)]]P_n^\lambda(x,\varphi)=(n+1)P_{n+1}^\lambda(x,\varphi)\\&[n+\lambda-\mathrm{i}x-(\lambda-\mathrm{i}x)\exp(\mathrm{i}\partial_x)]P_n^\lambda(x,\varphi)=(2\lambda+n-1)\exp(-\mathrm{i}\varphi)P_{n-1}^\lambda(x,\varphi).\end{aligned} \tag{2.9}$$

3. The Hahn polynomials of an imaginary argument

Just as in the case of the Pollaczek polynomials we shall introduce the Hahn polynomials of an imaginary argument (Suslov 1984a):

$$p_n(x)=p_n^{(\alpha,\beta)}(x,\gamma)=\mathrm{i}^{-n}h_n^{(\alpha,\beta)}(z,N) \tag{3.1}$$

where $z=\frac{1}{2}\mathrm{i}(x+\gamma)-\frac{1}{2}(\beta+1)$ and $N=-\frac{1}{2}(\alpha+\beta)+\mathrm{i}\gamma$. The definition (3.1) deals with the Hahn polynomials $h_n^{(\alpha,\beta)}(z,N)$, analytically continued both in variable z and parameter N to the complex plane (for their properties see, for instance, Nikiforov *et al* 1984).

Taking into account the three-term recurrence relation for the Hahn polynomials $h_n^{(\alpha,\beta)}(z,N)$ and formula (3.1) we find that

$$\begin{aligned}xp_n(x)&=\frac{2(n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}p_{n+1}(x)+\frac{\gamma(\beta^2-\alpha^2)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}p_n(x)\\&\quad+\frac{2(\alpha+n)(\beta+n)\{\gamma^2+[n+\frac{1}{2}(\alpha+\beta)]^2\}}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}p_{n-1}(x)\\&\quad p_0(x)=1,\qquad p_{-1}(x)=0.\end{aligned} \tag{3.2}$$

Therefore the polynomials $p_n^{(\alpha,\beta)}(x,\gamma)$ have the real coefficients for the real values of the parameters α, β and γ.

† We note the following properties of the function $\Gamma(z)$ (see, for example, Abramowitz and Stegun 1964):

$$\Gamma^*(z)=\Gamma(z^*)\qquad \lim_{|x|\to\infty}(2\pi)^{-1/2}|\Gamma(\lambda+\mathrm{i}x)|\exp(\pi|x|/2)|x|^{-\lambda+1/2}=1.$$

The symbol * denotes complex conjugation.

As in the case of the Pollaczek polynomials, for the Hahn polynomials of an imaginary argument it is possible to prove the orthogonality relation with respect to a continuous measure.

Orthogonality property. For the real values of the parameters α, β and γ ($\alpha, \beta > -1$) the polynomials $p_n^{(\alpha,\beta)}(x, \gamma)$ are orthogonal in the infinite interval $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} p_m^{(\alpha,\beta)}(x, \gamma)p_n^{(\alpha,\beta)}(x, \gamma)\rho(x)\, \mathrm{d}x = d_n^2\delta_{mn} \tag{3.3}$$

with the weight

$$\rho(x) = \frac{1}{4\pi}\left|\Gamma\left(\mathrm{i}\frac{x-\gamma}{2}+\frac{\alpha+1}{2}\right)\Gamma\left(\mathrm{i}\frac{x+\gamma}{2}+\frac{\beta+1}{2}\right)\right|^2. \tag{3.3a}$$

Proof. Let us calculate the integral of the product of two Hahn polynomials $h_n(z) = h_n^{(\alpha,\beta)}(z, N)$:

$$I = \frac{1}{2\pi\mathrm{i}}\int_C h_m(z)h_n(z)\tilde{\rho}(z)\, \mathrm{d}z \tag{3.4}$$

where $\tilde{\rho}(z) = \Gamma(\beta+z+1)\Gamma(z-N+1)\Gamma(\alpha+N-z)\Gamma(-z)$, over some contour C, which separates the poles of the expressions $\Gamma(\beta+z+1)\Gamma(z-N+1)$ and $\Gamma(\alpha+N-z)\Gamma(-z)$ (figure 2). Using the symmetry properties of the Hahn polynomials (Suslov 1984a)

$$h_n^{(\alpha,\beta)}(z, N) = (-1)^n h_n^{(\beta,\alpha)}(N-z-1, N) = h^{(-N,\alpha+\beta+N)}(z-\alpha-N, -\alpha)$$

and the representation for these polynomials through the hypergeometric function (Nikiforov *et al* 1984)

$$h_n^{(\alpha,\beta)}(z, N) = \frac{(-1)^n(\beta+1)_n\Gamma(N)}{n!\Gamma(N-n)}\,{}_3F_2\left(\begin{matrix}-n, \alpha+\beta+n+1, -z\\ \beta+1, 1-N\end{matrix}\middle|1\right) \tag{3.5}$$

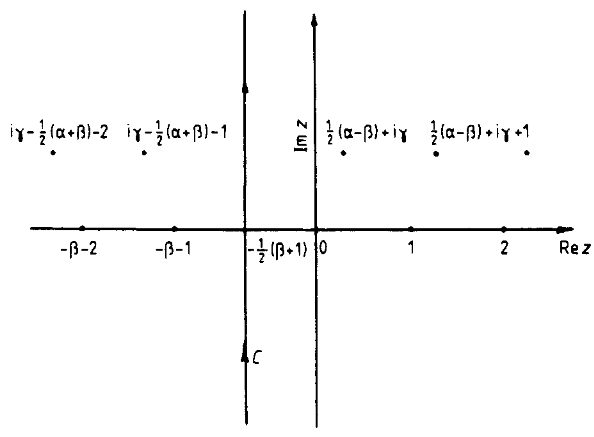

Figure 2. The contour $C(z = \frac{1}{2}\mathrm{i}(x+\gamma) - \frac{1}{2}(\beta+1), -\infty < x < \infty)$, on which the Hahn polynomials of an imaginary argument $p_n^{(\alpha,\beta)}(x, \gamma)$ are orthogonal, passes between the poles of the expressions $\Gamma(\beta+z+1)\Gamma(z-N+1)$ and $\Gamma(\alpha+N-z)\Gamma(-z)$ $(N = \mathrm{i}\gamma - \frac{1}{2}(\alpha+\beta); \alpha, \beta > -1)$.

we get

$$I=(-1)^m\frac{(\alpha+1)_m(\alpha+1)_n}{m!n!}$$
$$\times\sum_{k,k'}\frac{(-m)_k(\alpha+\beta+m+1)_k(-n)_{k'}(\alpha+\beta+n+1)_{k'}\Gamma(1-N+m)\Gamma(\alpha+\beta+N+n+1)}{(\alpha+1)_k k!(\alpha+1)_{k'}k'!\Gamma(1-N+k)\Gamma(\alpha+\beta+N+k'+1)}$$
$$\times\frac{1}{2\pi\mathrm{i}}\int_C\Gamma(\beta+z+1)\Gamma(z-N+k+1)\Gamma(\alpha+N-z+k')\Gamma(-z)\,\mathrm{d}z. \tag{3.6}$$

For $\alpha,\beta>-1$ the Barnes' lemma (see Whittaker and Watson 1927) and the known integral representation for the B function allow us to prove the identity

$$\frac{1}{2\pi\mathrm{i}}\int_C\Gamma(\beta+z+1)\Gamma(z-N+k+1)\Gamma(\alpha+N-z+k')\Gamma(-z)\mathrm{d}z$$
$$=2^{-(\alpha+\beta+1)}\Gamma(\alpha+\beta+N+k'+1)\Gamma(-N+k+1)$$
$$\times\int_{-1}^{1}\left(\frac{1-s}{2}\right)^{k+k'}(1-s)^\alpha(1+s)^\beta\,\mathrm{d}s. \tag{3.7}$$

Therefore the formula (3.6) takes the form

$$I=(-1)^m\Gamma(1-N+m)\Gamma(\alpha+\beta+N+n+1)\frac{(\alpha+1)_m(\alpha+1)_n}{m!n!}$$
$$\times\int_{-1}^{1}\mathrm{d}s(1-s)^\alpha(1+s)^\beta F\left(-m,\alpha+\beta+m+1,\alpha+1;\frac{1-s}{2}\right)$$
$$\times F\left(-n,\alpha+\beta+n+1,\alpha+1;\frac{1-s}{2}\right). \tag{3.8}$$

Taking into account the expression for the Jacobi polynomials $P_n^{(\alpha,\beta)}(s)$ through the hypergeometric function $F(\alpha,\beta,\gamma;z)$ (see, for instance, Bateman and Erdelyi 1953), for the integral (3.4) from (3.8) we find

$$\frac{1}{2\pi\mathrm{i}}\int_C h_m(z)h_n(z)\tilde{\rho}(z)\,\mathrm{d}z$$
$$=(-1)^m2^{-(\alpha+\beta+1)}\Gamma(1-N+m)\Gamma(\alpha+\beta+N+1)$$
$$\times\int_{-1}^{1}P_m^{(\alpha,\beta)}(s)P_n^{(\alpha,\beta)}(s)(1-s)^\alpha(1+s)^\beta\,\mathrm{d}s. \tag{3.9}$$

In the formula (3.9) we will choose a contour C in such a way that $z=\frac{1}{2}\mathrm{i}(x+\gamma)-\frac{1}{2}(\beta+1)$. Then according to the equalities (3.1) and (3.9), the orthogonality property (3.3) of the Hahn polynomials of an imaginary argument $p_n^{(\alpha,\beta)}(x,\gamma)$ follows from the orthogonality of the Jacobi polynomials $P_n^{(\alpha,\beta)}(s)$ for $\alpha,\beta>-1$. Besides, the formula (3.9) leads to the next value of the square of the polynomials $p_n^{(\alpha,\beta)}(x,\gamma)$ norm:

$$d_n^2=\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)|\Gamma(\frac{1}{2}(\alpha+\beta)+\mathrm{i}\gamma+n+1)|^2}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}. \tag{3.10}$$

In the same way it is possible to introduce the polynomials Askey and Wilson 1982, Suslov 1984a)

$$q_n^{(\alpha)}(x,\delta)=p_n^{(\alpha,\alpha)}(x,-\mathrm{i}\delta)=\mathrm{i}^{-n}h_n^{(\alpha,\alpha)}(z,N) \tag{3.11}$$

where $z=\frac{1}{2}ix-\frac{1}{2}(\alpha-\delta+1)$ and $N=\delta-\alpha$, which takes real values for the real parameters α and δ and variable x. Using the same considerations *mutatis mutandis* (the location of the poles of the Γ functions is given in figure 3), we come to the orthogonality relation for the polynomials $q_n^{(\alpha)}(x,\delta)$:

$$\int_{-\infty}^{\infty} q_m^{(\alpha)}(x,\delta)q_n^{(\alpha)}(x,\delta)\rho(x)\,\mathrm{d}x=d_n^2\delta_{mn} \qquad (\alpha>-1,|\delta|<\alpha+1). \tag{3.12}$$

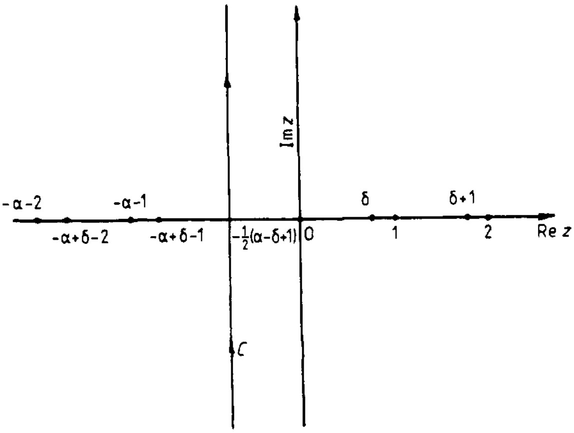

Figure 3. The contour C ($z=\frac{1}{2}ix-\frac{1}{2}(\alpha-\delta+1)$, $-\infty<x<\infty$), on which the Hahn polynomials of an imaginary argument $q_n^{(\alpha)}(x,\delta)$ are orthogonal, passes between the poles of the expressions $\Gamma(\alpha+z+1)\Gamma(z-N+1)$ and $\Gamma(\alpha+N-z)\Gamma(-z)$ ($N=\delta-\alpha$, $\alpha>-1$, $|\delta|<\alpha+1$).

Here

$$\rho(x)=\frac{1}{4\pi}\left|\Gamma\left(i\frac{x}{2}+\frac{\alpha+\delta+1}{2}\right)\Gamma\left(i\frac{x}{2}+\frac{\alpha-\delta+1}{2}\right)\right|^2 \tag{3.12a}$$

and

$$d_n^2=\frac{\Gamma^2(\alpha+n+1)\Gamma(\alpha+\delta+n+1)\Gamma(\alpha-\delta+n+1)}{n!(2\alpha+2n+1)\Gamma(2\alpha+n+1)}.$$

We would also like to mention that the weights for the Hahn polynomials of an imaginary argument (3.3a) and (3.12a), as well as the weight for the Pollaczek polynomials (2.7), satisfy the closedness criterion (see Nikiforov and Ouvarov 1983). Therefore these polynomials form the closed orthogonal systems of functions.

The Hahn polynomials of an imaginary argument $p_n^{(\alpha,\beta)}(x,\gamma)$ and $q_n^{(\alpha)}(x,\delta)$ are closely related to the unitary irreducible representations of the Lorentz group SO(3, 1) for the principal and complementary series, respectively (see § 5).

According to the relations (3.1) and (3.11) the polynomials $p_n^{(\alpha,\beta)}(x,\gamma)$ and $q_n^{(\alpha)}(x,\delta)$ satisfy a difference equation, Rodrigues' formula, etc, being true for them owing to the known properties of the Hahn polynomials $h_n^{(\alpha,\beta)}(z,N)$. We note, for instance, the symmetry relations:

$$p_n^{(\alpha,\beta)}(x,\gamma)=p_n^{(\beta,\alpha)}(x,-\gamma)=(-1)^n p_n^{(\beta,\alpha)}(-x,\gamma)$$
$$q_n^{(\alpha)}(x,\delta)=q_n^{(\alpha)}(x,-\delta)=(-1)^n q_n^{(\alpha)}(-x,\delta).$$

On the other hand, from the orthogonality relations (3.3) and (3.12) it is possible to come to the discrete orthogonality property of the corresponding Hahn polynomials $h_n^{(\alpha,\beta)}(z, N)$ as a result of the analytic continuation.

Consider an analogue of the Legendre polynomials: the polynomials $t_n(x, \gamma) = p_n^{(0,0)}(x, \gamma)$, which is natural to call the Tchebichef polynomials of an imaginary argument. Their weight $\rho(x)$ and squared norm d_n^2 are equal to

$$\rho(x) = \frac{\pi}{2(\cosh \pi x + \cosh \pi\gamma)} \qquad d_n^2 = \frac{\pi\gamma}{(2n+1)\sinh \pi\gamma} \prod_{k=0}^{n-1} [(n-k)^2 + \gamma^2].$$

It is interesting to note that in this case the weight $\rho(x)$ satisfies along with the difference equation the nonlinear differential equation

$$\rho'(x) + (2\sinh \pi x)\rho^2(x) = 0. \tag{3.13}$$

We also mention the polynomials $q_n^{(0)}(x, \delta)$, for which

$$\rho(x) = \frac{\pi}{2(\cosh \pi x + \cos \pi\,\delta)}$$

$$d_n^2 = \frac{\pi\delta}{(2n+1)\sin \pi\delta} \prod_{k=0}^{n-1} [(n-k)^2 - \delta^2] \qquad (|\delta| < 1).$$

The Hahn polynomials of an imaginary argument $q_n^{(0)}(x, 1/2)$ and the Pollaczek polynomials $P_n^{(1/2)}(x, \pi/2)$ are orthogonal on the interval $(-\infty, \infty)$ with the same weight $\rho(x) = \text{constant}/\cosh \pi x$. As a consequence of this they are related by the equality

$$q_n^{(0)}(x, 1/2) = (\tfrac{1}{2})_n P_n^{(1/2)}(x, \pi/2). \tag{3.14}$$

4. The Pollaczek polynomials in a quasipotential model of the relativistic oscillator

For the consistent three-dimensional description of a relativistic two-particle system in quantum field theory the quasipotential approach has been formulated. Some relativistic generalisations of the known exactly solvable problems of quantum mechanics have been considered in the framework of this approach (Logunov and Tavkhelidze 1963, Kadyshevsky 1968, Kadyshevsky and Mateev 1968). Thus, in the papers by Atakishiyev *et al* (1980) and Atakishiyev (1983, 1984) a model of the linear oscillator in the relativistic configurational x representation (Kadyshevsky *et al* 1968) has been studied, which is described by the difference Hamiltonian:

$$H(x) = mc^2 \cosh(\mathrm{i}\lambda\partial_x) + \tfrac{1}{2}m\omega^2 x(x+\mathrm{i}\lambda)\exp(\mathrm{i}\lambda\partial_x) \tag{4.1}$$

where $\lambda = \hbar/mc$ is the Compton wavelength, $\partial_x = \mathrm{d}/\mathrm{d}x$ and $\exp(\alpha\partial_x)f(x) = f(x+\alpha)$. We shall prove further that the square-integrable solutions of the equation

$$H(x)\psi_n(x) = E_n\psi_n(x) \tag{4.2}$$

are expressed through the Pollaczek polynomials $P_n^{\nu}(x, \varphi)$ with the fixed value of the parameter $\varphi = \pi/2$. In fact, having separated the factors $[\nu(\nu-1)]^{-\mathrm{i}x/2\lambda}$ and $\Gamma(\nu + \mathrm{i}x/\lambda)$, $\nu = \frac{1}{2} + [\frac{1}{4} + (c/\lambda\omega)^2]^{1/2}$, which determine the asymptotic behaviour of the wavefunction $\psi_n(x)$ at the points $x=0$ and $x=\infty$, respectively, we represent it in the form

$$\Psi_n(x)=C_n[\nu(\nu-1)]^{-ix/2\lambda}\Gamma(\nu+ix/\lambda)\Omega(x). \tag{4.3}$$

The substitution of (4.3) in (4.2) leads to the difference equation for $\Omega(x)$:

$$[(\nu-ix/\lambda)\exp(i\lambda\partial_x)+(\nu+ix/\lambda)\exp(-i\lambda\partial_x)]\Omega(x)=2(E_n/\hbar\omega)\Omega(x). \tag{4.4}$$

Comparison of this equation with (2.8) shows that the square-integrable solutions of (4.4) are the polynomials $\Omega(x)=P_n^\nu(x/\lambda, \pi/2)$, while the corresponding eigenvalues of the Hamiltonian (4.1) are equal to $E_n=\hbar\omega(n+\nu)$, $n=0,1,2,\ldots$. The orthonormality of the wavefunctions

$$\begin{aligned}\Psi_n(x)&=C_n[\nu(\nu-1)]^{-ix/2\lambda}\Gamma(\nu+ix/\lambda)P_n^\nu(x/\lambda, \pi/2)\\ C_n&=2^\nu[n!/2\pi\lambda\Gamma(n+2\nu)]^{1/2}\end{aligned} \tag{4.5}$$

is the consequence of the continuous orthogonality relation (2.6) for the Pollaczek polynomials.

As is known (Atakishiyev *et al* 1980) a dynamical symmetry group for the oscillator with the Hamiltonian (4.1) is the group SU(1, 1) (or isomorphic groups $SO(2,1)\sim Sp(2,R)\sim SL(2,R)$), whose generators are realised by the difference operators

$$K_0=\frac{1}{\hbar\omega}H(x) \qquad K_\pm=\frac{x}{\lambda}\pm iK_0\mp\frac{ic}{\lambda\omega}\exp(-i\lambda\partial_x). \tag{4.6}$$

It is easy to see that their action on the eigenfunctions of the Hamiltonian (4.1), i.e.

$$\begin{aligned}&K_-\psi_n(x)=\kappa_n\psi_{n-1}(x) \qquad K_+\psi_n(x)=\kappa_{n+1}\psi_{n+1}(x)\\ &\kappa_n=[n(n+2\nu-1)]^{1/2}\end{aligned} \tag{4.7}$$

follows from the formulae (2.9). The wavefunctions $\psi_n(x)$ (see (4.5)) are the basis functions of the infinite-dimensional irreducible unitary representation $D^+(\nu)$ (discrete positive series) of the universal covering group $\widetilde{SU}(1,1)$. Therefore, the relations (4.5) lead to the group-theoretic interpretation for the main properties of the Pollaczek polynomials.

Taking into account the limiting formula for the Pollaczek polynomials (Pollaczek 1950b), i.e.

$$\lim_{\nu\to\infty}\nu^{-n/2}P_n^\nu\left(\frac{\nu^{1/2}x-\nu\cos\varphi}{\sin\varphi}, \varphi\right)=\frac{1}{n!}H_n(x)$$

where $H_n(x)$ are the Hermite polynomials, it is easy to show from (4.5) that in the limit when the velocity of light c tends to infinity, $\psi_n(x)$ coincide with the wavefunctions of the non-relativistic linear oscillator.

5. The Hahn polynomials of an imaginary argument and representations of the Lorentz group SO(3, 1)

In the present section we discuss the close relationship between unitary irreducible representations of the Lorentz group SO(3, 1) and the Hahn polynomials of an imaginary argument introduced above. In this way one manages to state the basic facts from the representation theory of the Lorentz group (Gel'fand *et al* 1958, Naimark 1964) in the form which is close to the well known problem of the coupling of two momenta in quantum mechanics.

Let us consider Minkowski space, i.e. the four-dimensional real pseudo-Euclidean space, in which a distance (interval) is determined by the quadratic form

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

All the possible rotations in the three-dimensional space (x_1, x_2, x_3) and boosts, i.e. the hyperbolic rotations in the planes (x_0, x_1), (x_0, x_2) and (x_0, x_3), form the proper Lorentz group SO(3, 1).

Let $\boldsymbol{J}$ and $\boldsymbol{K}$ be the infinitesimal operators of rotations and boosts, respectively, which satisfy the commutation relations

$$\begin{aligned} &[J_p, J_q] = \mathrm{i} e_{pqr} J_r \qquad &&[J_p, K_q] = \mathrm{i} e_{pqr} K_r \\ &[K_p, K_q] = -\mathrm{i} e_{pqr} J_r &&(p, q, r = 1, 2, 3) \end{aligned} \tag{5.1}$$

where e_{pqr} is the Levi-Civita symbol.

The transformation

$$\boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{J} + \mathrm{i}\boldsymbol{K}) \qquad \boldsymbol{B} = \tfrac{1}{2}(\boldsymbol{J} - \mathrm{i}\boldsymbol{K})$$

leads to the commutation relations for two independent angular momenta:

$$\begin{aligned} &[A_p, A_q] = \mathrm{i} e_{pqr} A_r \qquad &&[B_p, B_q] = \mathrm{i} e_{pqr} B_r \\ &[A_p, B_q] = 0 &&(p, q, r = 1, 2, 3). \end{aligned} \tag{5.2}$$

Therefore the construction of irreducible representations of the group SO(3, 1) is closely connected with the problem of the coupling of two 'complex conjugate' momenta $\boldsymbol{A}$ and $\boldsymbol{B}$ into one 'real' vector $\boldsymbol{J} = \boldsymbol{A} + \boldsymbol{B}$. For the unitary representations we have†

$$\boldsymbol{J}^+ = \boldsymbol{J} \qquad \boldsymbol{K}^+ = \boldsymbol{K} \qquad \boldsymbol{A}^+ = \boldsymbol{B}.$$

(The symbol L^+ denotes the Hermitian conjugation of an operator L.)

According to the commutation rules (5.2) in a space of irreducible representation of the group SO(3, 1) it is possible to construct the basis $\Phi_{m_1 m_2}$, on which the operators $A_\pm = A_1 \pm \mathrm{i} A_2$, A_3 and $B_\pm = B_1 \pm \mathrm{i} B_2$, B_3 act by the formulae

$$\begin{aligned} &A_\pm \Phi_{m_1 m_2} = [(j_1 \mp m_1)(j_1 \pm m_1 \pm 1)]^{1/2} \Phi_{m_1 \pm 1, m_2} \qquad &&A_3 \Phi_{m_1 m_2} = m_1 \Phi_{m_1 m_2} \\ &B_\pm \Phi_{m_1 m_2} = [(j_2 \mp m_2)(j_2 \pm m_2 + 1)]^{1/2} \Phi_{m_1, m_2 \pm 1} &&B_3 \Phi_{m_1 m_2} = m_2 \Phi_{m_1 m_2}. \end{aligned} \tag{5.3}$$

In the case of the Lorentz group the constants j_1, j_2, m_1, m_2 take some complex values. Since $\boldsymbol{A}^+ = \boldsymbol{B}$, then $j_1^* = j_2$ and $m_1^* = m_2$. Vectors $\Phi_{m_1 m_2}$ are the eigenvectors of two Hermitian operators $J_3 = A_3 + B_3$ and $K_3 = \mathrm{i}^{-1}(A_3 - B_3)$:

$$J_3 \Phi_{m_1 m_2} = m \Phi_{m_1 m_2} \qquad K_3 \Phi_{m_1 m_2} = \lambda \Phi_{m_1 m_2} \tag{5.4}$$

and they correspond to the real eigenvalues $m = m_1 + m_2$ and $\lambda = \mathrm{i}^{-1}(m_1 - m_2)$. Therefore

$$m_1 = \tfrac{1}{2}(m + \mathrm{i}\lambda) \qquad m_2 = \tfrac{1}{2}(m - \mathrm{i}\lambda). \tag{5.5}$$

† The Lorentz group SO(3, 1) is non-compact and its irreducible representations are infinite-dimensional. The correct definition of representations for this case can be found, for example, in Naimark (1964).

For the basis $\chi_{\lambda m} \equiv \Phi_{m_1 m_2}$, where the quantum numbers are connected by the formula (5.5), the orthogonality and normalisation relations are valid:

$$(\chi_{\lambda m}|\chi_{\lambda' m'}) = \delta_{mm'}\delta(\lambda - \lambda'). \tag{5.6}$$

Here $\delta(\xi)$ is the Dirac δ function.

On the other hand, the infinitesimal operators $\boldsymbol{J}$ satisfy the commutation relations of the angular momentum. Therefore in a space of irreducible representation there exists the basis Ψ_{jm}, on which the operators $J_\pm = J_1 \pm \mathrm{i}J_2$ and J_3 act by the formulae

$$J_\pm \Psi_{jm} = [(j \mp m)(j \pm m + 1)]^{1/2}\Psi_{j,m\pm1} \qquad J_3\psi_{jm} = m\psi_{jm}. \tag{5.7}$$

Here j is an integer or half-integer positive number and $m = -j, -j+1, \ldots, j-1, j$.

From the infinitesimal point of view the study of irreducible representations of the Lorentz group SO(3, 1) is reduced to defining the form of the operators $K_\pm = K_1 \pm \mathrm{i}K_2$ and K_3 in the basis Ψ_{jm} (Gel'fand *et al* 1958, Naimark 1964).

To find out how the operators $K_\pm$ and K_3 act on the basis Ψ_{jm}, we expand the vector Ψ_{jm} over the eigenfunctions of the operator K_3:

$$\Psi_{jm} = \int_{-\infty}^{\infty} \mathrm{d}\lambda \langle m_1 m_2|jm\rangle \Phi_{m_1 m_2} \tag{5.8}$$

with m_1 and m_2 defined in (5.5), and then determine the coefficients of the expansion (5.8). Acting by the operators $J_\pm = A_\pm + B_\pm$ on both sides of the equality (5.8) and taking into account (5.7), (5.3) and (5.6), we obtain the recurrence relations for these coefficients, which are well known in the theory of angular momenta:

$$\begin{aligned}
&[(j \mp m)(j \pm m + 1)]^{1/2}\langle m_1 m_2|j, m \pm 1\rangle \\
&\qquad = [(j_1 \pm m_1)(j_1 \mp m_1 + 1)]^{1/2}\langle m_1 \mp 1, m_2|jm\rangle \\
&\qquad + [(j_2 \pm m_2)(j_2 \mp m_2 + 1)]^{1/2}\langle m_1, m_2 \mp 1|jm\rangle.
\end{aligned} \tag{5.9}$$

The substitution

$$\langle m_1 m_2|jm\rangle = \alpha_\pm C^\pm_{m_1 m_2 m} u^\pm_{jm}(m_1) \tag{5.10}$$

where

$$C^\pm_{m_1 m_2 m} = \left(\frac{\Gamma(j_1 + m_1 + 1)\Gamma(j_2 + m_2 + 1)(j - m)!}{\Gamma(j_1 - m_1 + 1)\Gamma(j_2 - m_2 + 1)(j + m)!}\right)^{\pm 1/2}$$

and

$$\alpha_+^{-1} = \sin \pi(j_1 - m_1 + 1) \qquad \alpha_-^{-1} = \sin \pi(j_2 + m_2 + 1)$$

leads to simple difference-recurrence formulae for the functions $u^\pm_{jm}(m_1)$:

$$u^+_{j,m+1}(m_1) = \nabla u^+_{jm}(m_1) \qquad u^-_{j,m-1}(m_1) = \Delta u^-_{jm}(m_1) \tag{5.11}$$

where $\Delta f(x) = f(x+1) - f(x)$ and $\nabla f(x) = f(x) - f(x-1)$. From (5.11) exactly in the same manner as in deducing the general expression for the Clebsch-Gordan coefficients (Nikiforov *et al* 1984) we get

$$\begin{aligned}
\langle m_1 m_2|jm\rangle = &\frac{(-1)^{j-m}}{\sin \pi(j_1 - m_1 + 1)} \frac{A}{(2j)!}\left(\frac{\Gamma(j_1 - m_1 + 1)\Gamma(j_2 - m_2 + 1)(j + m)!}{\Gamma(j_1 + m_1 + 1)\Gamma(j_2 + m_2 + 1)(j - m)!}\right)^{1/2} \\
&\times \Delta^{j-m}_{m_1}\left(\frac{\Gamma(j_1 + m_1 + 1)\Gamma(j_2 + j - m_1 + 1)}{\Gamma(j_1 - m_1 + 1)\Gamma(j_2 - j + m_1 + 1)}\right).
\end{aligned} \tag{5.12}$$

Here A is a constant, determined by the normalisation condition

$$\|\psi_{jj}\|^2=\int_{-\infty}^{\infty} d\lambda |\langle m_1 m_2|jj\rangle|^2=1. \tag{5.13}$$

Using the Barnes' lemma (Whittaker and Watson 1927) it is easy to check that for the unitary irreducible representations of the Lorentz group SO(3, 1) the condition (5.13) is satisfied†:

(a) in the case of the principal series, when

$$j_1=j_2^*=\tfrac{1}{2}(\mu-1+\mathrm{i}\gamma) \qquad j=|j_1+j_2+1|, |j_1+j_2+1|+1, \ldots$$

(μ is an integer or half-integer and γ is an arbitrary real number);

(b) for the complementary series, when

$$j=j_2^*=\tfrac{1}{2}(\delta-1) \qquad j=j_1-j_2, j_1-j_2+1, \ldots$$

(δ is a real number and $|\delta|<1$). For both cases

$$\frac{2|A|}{\sqrt{\pi}(2j)!}=\left(\frac{2j+1}{\Gamma(j+j_1-j_2+1)\Gamma(j-j_1+j_2+1)\Gamma(j-j_1-j_2)\Gamma(j+j_1+j_2+2)}\right)^{1/2}. \tag{5.14}$$

For our further consideration it is convenient to put $A=\mathrm{i}^{j-m}|A|$.

By analogy with the theory of angular momenta coefficients of the expansion (5.8) are called 'complexificated Clebsch-Gordan coefficients' (Smorodinsky and Shepelev 1971).

Using the Rodrigues' formula for the Hahn polynomials (see, for example, Nikiforov *et al* 1984), in accordance with the relations (3.1), (3.11) and (5.12), the complexificated Clebsch-Gordan coefficients can be expressed through the Hahn polynomials of an imaginary argument. For the principal series we have

$$\langle m_1 m_2|jm\rangle=f\frac{(\rho(\lambda))^{1/2}}{d_{j-m}}p_{j-m}^{(m-\mu,m+\mu)}(\lambda,\gamma) \tag{5.15}$$

where $\rho(\lambda)$ and d_n are the weight and norm of the polynomials $p_n^{(\alpha,\beta)}(x,\gamma)$ (see § 3). In the case of the complementary series

$$\langle m_1 m_2|jm\rangle=f\frac{(\rho(\lambda))^{1/2}}{d_{j-m}}q_{j-m}^{(m)}(\lambda,\delta). \tag{5.16}$$

In the formulae (5.15) and (5.16) there appears the factor

$$f=\left(\frac{\sin\pi(j_2-m_2+1)}{\sin\pi(j_1-m_1+1)}\right)^{1/2} \qquad ff^*=1.$$

Up to this factor the complexificated Clebsch-Gordan coefficients are real.

The formulae (5.8), (5.15) and (5.16) allow us to derive the matrix elements of the operators $K_\pm$ and K_3 in the basis Ψ_{jm}, thus exploiting the properties of the Hahn polynomials of an imaginary argument studied above. For instance, using the three-term recurrence relation (3.2) we obtain

$$K_3\Psi_{jm}=a_{jm}\Psi_{j-1,m}+b_{jm}\Psi_{jm}+a_{j+1,m}\Psi_{j+1,m} \tag{5.17}$$

† We note that the usually used notations (Gel'fand *et al* 1958) are connected with ours in the following way: $l_0=\mu$, $l_1=\mathrm{i}\gamma$, $l=j$, $\xi_{lm}=\mathrm{i}^{j-m}\Psi_{jm}$.

where for the principal series

$$a_{jm} = \left(\frac{(j^2-m^2)(j^2-\mu^2)(j^2+\gamma^2)}{(2j-1)j^2(2j+1)}\right)^{1/2} \qquad b_{jm} = \frac{m\mu\gamma}{j(j+1)}. \tag{5.18}$$

In the case of the complementary series in the formulae (5.17) and (5.18) it is necessary to set $\mu = 0$ and $\gamma = -\mathrm{i}\delta$.

Using (5.7), (5.17) and the commutation relations (5.1) it is easy to determine the action of the operators $K_\pm$ on the basis Ψ_{jm}. Thus, the action of the infinitesimal operators $\boldsymbol{J}$ and $\boldsymbol{K}$ on the basis Ψ_{jm} of the irreducible representation of the Lorentz group SO(3, 1) is obtained with the help of the previously studied properties of the Hahn polynomials of an imaginary argument. Conversely, from the formulae, determining the action of the operators $\boldsymbol{J}$ and $\boldsymbol{K}$ on the basis Ψ_{jm}, the group-theoretic interpretation for the main properties of the Hahn polynomials of an imaginary argument arise naturally.

The reasonings carried out by us allow to obtain simple expressions for the boost matrix elements of the Lorentz group SO(3, 1):

$$d^{(\mu,\gamma)}_{jj'm}(t) = (\Psi_{jm}|\exp(-\mathrm{i}tK_3)\Psi_{j'm}).$$

In the case of the principal series (μ, γ) the formulae (5.8) and (5.15) lead to the following integral representation (Smorodinsky and Shepelev 1971, Suslov 1982, 1984a):

$$d^{(\mu,\gamma)}_{jj'm}(t) = \int_{-\infty}^{\infty} \bar{p}_{j-m}(\lambda)\exp(-\mathrm{i}t\lambda)\bar{p}_{j'-m}(\lambda)\rho(\lambda)\,\mathrm{d}\lambda \tag{5.19}$$

where $\bar{p}_{j-m}(\lambda) = d^{-1}_{j-m}p^{(m-\mu,m+\mu)}_{j-m}(\lambda, \gamma)$. An analogous integral representation for the boost matrix in the case of the complementary series is easily derived from (5.8) and (5.16).

The integral representation (5.19) allows us to study the function $d^{(\mu,\gamma)}_{jj'm}(t)$ on the basis of the known properties of the Hahn polynomials of an imaginary argument. In particular, owing to the orthogonality property (3.3) we get

$$d^{(\mu,\gamma)}_{jj'm}(0) = \delta_{jj'}.$$

In conclusion we note that since the Lorentz group SO(3, 1) is a dynamical symmetry group for the non-relativistic Coulomb problem in the case of a continuous spectrum, in this problem the Hahn polynomials of an imaginary argument also arise (Suslov 1984b).

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